Modeling expected inventory order crossovers

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Abstract

Expected inventory order crossovers occur if at the moment of ordering it is expected that orders will not arrive in the sequence they are ordered. Recent research has shown that (a) expected inventory order crossovers will be encountered more frequently in the future, and that (b) use of a myopic order-up-to policy based on a stochastic dynamic programming approach leads to improved performance compared to the classical approach. In this paper we show that the improved policy is still heuristic in nature, as it neglects several control options that are available on the various ordering moments and makes some restrictive assumptions with respect to the separability (i.e., decomposability) of the stochastic dynamic programming problem. We propose further improvements in the policy for situations where a quadratic cost function is appropriate. Finally, we present some insights in the determination of safety stock levels in case of expected inventory order crossovers.

Keywords:
Inventory management; Control policy; Mathematical modeling

1 INTRODUCTION

Order crossovers occur if orders do not arrive in the same sequence as they were ordered. This phenomenon is caused by differences in the lead-times of the orders. Order crossovers are often neglected in inventory models. Whenever they are taken into account, most inventory models assume that lead-time differences are caused by a stochastic process (Figure 1). This paper focuses on order crossovers that are the result of dynamic processes instead of stochastic processes.

Figure 1 Two types of order crossovers

In dynamic processes, lead-time fluctuations that occur are known in advance and can be anticipated upon. Dynamic lead-time fluctuations may occur due to contract changes, expediting policies, dual-sourcing policies from different geographical areas, transportation mode changes, etcetera. Riezebos (2006) has shown that these dynamic variations will be encountered more frequently in the future. The main reason is that the above mentioned
instruments are increasingly used in modern supply chain management in order to increase flexibility (Robinson, Bradley, Thomas, 2001; Bradley and Robinson, 2005). However, the use of these instruments may lead to order crossovers. We denote such order crossovers as expected order crossovers, as they can be anticipated upon. Figure 1 illustrates the difference between both types of order crossovers.

This paper studies the consequences of expected order crossovers for a single-item inventory system, with known (variable) ordering moments (discrete-time, periodic review, with variable review periods). It is assumed that the future ordering moments are known in advance, as well as the lead-times at these moments. Order crossovers may occur due to dynamic fluctuations in both lead times and review periods. However, the stochastic variation in these parameters is considered to be minor and therefore neglected. Demand is stochastic and forecasts for future demand are available and may be updated at the next ordering moment.

For this problem, Gaalman and Riezebos (2005) showed that the standard order-up-to policy that is available in inventory management systems leads to incorrect decisions in case of expected order crossovers. They derived an improved myopic order-up-to policy based on a stochastic dynamic programming approach.

In this paper we show that the improved policy is still heuristic in nature, as it neglects several control options that are available on the various ordering moments and makes some restrictive assumptions with respect to the separability (i.e., decomposability) of the stochastic dynamic programming problem. We propose further improvements in the policy for situations where a quadratic cost function is appropriate. Finally, we present some insights in the determination of safety stock levels in case of expected inventory order crossovers.

2 ORDER CROSSES AND INVENTORY ORDERING POLICIES

The originating work on the optimal inventory equation defines an optimal ordering policy as a set of functions that specify how much to order in each of the \( J \) stages and yield minimum total expected costs (Bellmann, Glicksberg, Gross, 1955). The decision in stage \( j \) is based on a system state variable. In most inventory models, the stock level is used as system state variable, but Aviv (2003) notes that in general the system state variable can be considered as a multi-dimensional vector, consisting of various variables that might have impact on the current decision.

The standard periodic review order-up-to policy with stochastic demand, known ordering moments and known lead times is treated in all standard inventory modules of ERP systems. It uses the echelon inventory position at the moment of ordering as system state variable. Usually it is presented in textbooks (e.g. Silver, Pyke and Peterson 1998, Tersine 1988) with fixed review periods, but mathematically the formulation for variable review periods is similar:

\[
Q_j = \hat{D}_{o_j+j+1-o_j} + M_{j+1} - E_{o_j}
\]

where \( E_{o_j} = I_{o_j} + \sum_{l:o_j < o_l \land l \geq o_j} Q_l \) and

\[
E_{o_j} = E_{o_j-1} + Q_{j-1} - D_{o_{j-1},o_{j-1}}^{act}
\]

\[ Q_j = \text{Size of order } j, \text{ at order moment } o_j \]

\[ L_j = \text{Lead time of order } j \]

\[ M_t = \text{Minimum required stock just before time } t \]

\[ O = \{ o_j; j = 1, \ldots, J \} = \text{Ordered set of ordering moments} \{ o_j < o_{j+1} \} \]

\[ R = \{ r_j; j = 1, \ldots, J \} = \text{Set of arrival moments} \{ r_j = o_j + L_j \} \]

\[ E_t = \text{Echelon inventory position at time } t \]

\[ I_t = \text{Net on hand inventory at time } t \]

\[ D_{t,u}^{\text{act}} = \text{Actual demand from time } t \text{ to } t + u \]

\[ \hat{D}_{t,u}^s = \text{At time } s \text{ forecasted demand from time } t \text{ to } t + u \]

By convention, \( E_t \) represents the situation just before time \( t \), \( \hat{D}_{t,u}^s = \hat{D}_{t,l} + \hat{D}_{t+1,u-l} \) if \( u \geq l \), and \( \hat{D}_{t,u}^s = 0 \) if \( u \leq 0 \).

Formula (1) expresses the order size in terms of the difference between the total amount of items required before the next order that can be ordered arrives (at \( r_{j+1} \)): \( \hat{D}_{o_j,r_{j+1}-o_j} + M_{r_{j+1}} \) and the currently known amount of items that will become available to fulfill these requirements: \( E_{o_j} \). Hence it is an order-up-to policy (also known as a base stock policy), aiming at an end-of-period inventory equal to level \( M_{r_{j+1}} \), which we denote as the safety stock level. This level can either be determined simultaneously with the order size or it can be set in advance. Both cases will use characteristics of the demand distribution, i.e. the variance, and make a tradeoff between expected holding and stock-out costs. In the simultaneous model, the optimality of the order-up-to policy has been derived under restrictive cost and demand assumptions (Veinott, 1965). For instance it is assumed that the demand in each of the periods is independent and identically distributed (see Arrow, Karlin, Scarf, 1958) and that both demand and lot sizes are permitted to become negative.

Formula (2) shows that the system state variable \( E_{o_j} \) consists of two components: current on hand inventory available for future demand, and already released but not yet received orders \( (t : o_i < o_j \text{ and } r_i > o_j) \). The inventory balance equation (3) updates the echelon inventory position at \( o_j \) based on the position at \( o_{j-1} \), the order issued at the last ordering moment \( Q_{j-1} \) and the actual shipments \( D_{o_{j-1},o_j-1}^{\text{act}} \) between the two ordering moments.

A prerequisite of the standard ordering policy is that the arrival moment of the next replenishment is known. We restrict our attention to the must-ordering case, implicitly neglecting ordering costs. So, at each ordering moment an order is placed and the standard ordering policy can use the information on the arrival moment of the order placed at the next ordering moment.

The standard order-up-to policy can be characterized as a set of \( j=1..J \) linear functions of the system state variable \( E_{o_j} \). Although the decision made in a stage is only in effect for that stage (i.e. lot sizes may differ per stage), it has impact on the decisions in future stages. We therefore measure the impact of a set of \( J \) decisions on the total expected costs over the effective horizon.

Graves (1999), Aviv (2003) and Urban (2005) note that the optimality of the presented myopic order-up-to policy (see Johnson and Thompson, 1975) is only guaranteed in case of some strict assumptions. More realistic assumptions, such as auto-correlated demand, non-negative order sizes, and a demand rate that depends on the actual inventory level have impact on the optimality of this policy, although the impact might be small.
Gaalman and Riezebos (2005) show that in case of expected order crossovers the standard ordering policy leads to incorrect decisions that might have a strong impact on the total expected costs. They use the (originating) optimization problem and functional equation in order to deduce an improved myopic ordering policy. The main idea of this improved myopic policy is to focus on the system state at the moment of order arrival instead of order release. At the moment of ordering, this state has to be determined. In the deterministic case, this is a simple linear expression in known variables. However, if one or more variables are stochastic, the system state variable has to be estimated at the moment of ordering, based on the available information.

Based on the dynamic programming formulation, this paper will show that the improved myopic ordering policy is not optimal. We therefore first introduce the dynamic programming formulation and formulate the improved myopic ordering policy before we continue the discussion on the optimality of this policy.

In order to backwards determine the optimal solution to the dynamic programming problem over \( J \) stages, it is helpful to renumber the set of arrival moments \( A = \{a_i : i = 1,...,J\} \) into the ordered set \( A = \{a_i : i = 1,...,J\} \). The orders arrive during interval \([a_1,...,a_J]\), with \( a_1 = \min\{r_j\} \leq a_i \leq a_{i+1} \leq a_J = \max\{r_j\} \). The lot size arriving at \( a_i \) is denoted as \( Q_a \).

This lot size has been ordered \( L_i \) periods in advance, so there exists at least one ordering moment \( o_j \) for which \( o_j = a_i - L_i \). In this paper we assume that this ordering moment is unique. The lot size that is ordered at \( o_j \) is then either denoted as \( Q_{o_j} \) or \( Q_a \), as both are identical. This lot size arrives at arrival moment \( a_i \) and affects the inventory level just before the next arrival moment \( a_{i+1} \). Note that the order that arrives at \( a_j \) affects the end-of-period inventory just before the next order arrives, i.e. \( a_j+1 \). We therefore evaluate the effectiveness of the policy over the interval \([a_1,...,a_{j+1}]\).

The objective for the optimal lot size calculation is to minimize the inventory deviations from the order-up-to levels. These levels are not necessarily constant over time. They can be determined externally, based on managerial judgment. Alternatively, the levels can be based on the characteristics of the stochastic processes. Two options are available. Urban (2005) describes the first option. He determines the level for each stage separately within the inventory model, a prerequisite in the situation of auto-correlated demand he studies. The second option is somewhat in between. Aviv (2003) proposes installation-based order-up-to policies that include predetermined fixed safety stock as well as a variable demand forecast component that have to be taken into account within the inventory model.

For the dynamic programming formulation, a cost function is needed. We propose the cost function \( V(x) \) that penalizes a deviation of size \( x \), where \( V[0] = 0 \), \( V[x] > V[y] \Leftrightarrow \{x \geq y\} \), and \( V[x] = V[-x] \). The symmetrical property of \( V(x) \) is required, as otherwise we would not aim for the specified order-up-to level.

The optimal policy applied over \( J \) stages, starting with an initial stock of \( I_{a_i} \), will yield total expected costs equal to:

\[
G_t(I_{a_i}) = \min_{Q_a} E\left\{ \sum_{i=1}^{J} V[I_{a_i+1} - M_{a_{i+1}}] \right\} \tag{4}
\]

The expected total costs depend on the initial inventory level \( I_{a_i} \), as we have

\[
I_{a_{i+1}} = I_{a_i} + Q_a^e - D_{a_i,a_{i+1}} \tag{5}
\]
Formula (5) is the inventory balance equation based on the renumbered set of arrival moments and is equivalent with Formula (3). The inventory variable can be seen as a system state variable.

As long as demand is deterministic, we can easily determine the optimal control policy as:

\[ I_{a_{i+1}} = M_{a_{i+1}} \quad \forall i = 1..J \implies \]

\[ Q^a_i = D_{a_{i+1}}^{a_{i}} - a_i + M_{a_{i+1}} - I_i \quad \forall i = 1..J \]  

(6)

(7)

However, when demand is stochastic, two variables in the system equation (5) are not known at the time of taking the ordering decision: the state of the system at the arrival moment of \( a_i \) as well as the demand during the interval \([a_i, a_{i+1}]\).

If we assume the certainty-equivalence (CE) principle holds, then replacing the actual demand by its conditional expectation at the ordering moment results in a control policy similar to (7), denoted as the CE-policy or improved ordering policy.

The question is therefore whether the certainty-equivalence principle holds in case of stochastic demand. In order to answer this question, the optimal inventory problem is formulated as a backward stochastic dynamic programming problem by introducing the functional equations for \( G_{J}(I_{a_i}) : \)

\[ G_{j}(I_{a_i}) = E[V[I_{a_{j+1}} - M_{a_{j+1}}]^{Y_{j+1}}] \]

\[ G_{j-1-k}(I_{a_{j+k}}) = g_{j-1-k}(I_{a_{j+k}}) + G_{j+1-k}(I_{a_{j+1+k}}) \quad \forall k = 1..J - 1 \]

\[ g_{j-1-k}(I_{a_{j+k}}) = \min_{Q_j} E[V[I_{a_{j+1+k}} - M_{a_{j+1+k}}] + E_{g_{j+1-k}(I_{a_{j+1+k}})}]^{Y_{j+k}} \]

(8)

(9)

(10)

where the decision on \( Q_{j-1-k} \) is to be chosen such that the total expected costs over the remaining stages \( J - k, ..., J \) is minimal. The decision \( Q_{j-1-k} \) is based on the available information at the moment of ordering: \( Y_{j+k} \), but in the course of time additional information \( y_{j+1-k}, ..., y_J \) becomes available for decisions in the stages \( J + 1-k, ..., J \).

This problem is in general not easily solvable, but can be reduced to a simple and solvable case if the second term in (10): \( E_{g_{j+1-k}(I_{a_{j+1+k}})} \) has no effect on the optimal value of \( Q_{j-1-k} \). We denote this as separability, as the problem then reduces to a set of independent optimization problems

\[ g_{j-1-k}(I_{a_{j+k}}) = \min_{Q_j} E[V[I_{a_{j+1+k}} - M_{a_{j+1+k}}]^{Y_{j+k}}] \quad \forall k = 0..J - 1 \]

(11)

If it is allowed to apply separability, then the dynamic programming recursion reduces to (11), a set of independent optimization problems. The ‘optimal’ solution in this case is \( \hat{I}_{a_{i+1}} = M_{a_{i+1}} \forall i = 1,...,J \), with \( \hat{I}_i \) defined as the - at time \( s \)- estimated net on hand inventory level on time \( t \). Using conditional expectations we can write:

\[ \hat{I}_{a_{i+1}} = \hat{I}_{a_i} - L_i + Q^a_i - D_{a_{i+1}}^{a_{i}} \quad \forall i = 1,...,J \]

(12)

The improved ordering policy emerges, which can be described as:

\[ Q^a_i = D_{a_{i+1}}^{a_{i}} - a_i + M_{a_{i+1}} - \hat{I}_{a_i} \quad \forall i = 1,...,J \]

(13)

The system state variable in this policy is \( I^a_i \), which can be worked out using (12):

\[ \hat{I}_{a_{i+1}} = \hat{I}_{a_i} = I_i - D_{a_{i+1}} - a_i + \sum_{t:a_i \geq o_j, a_i < o_i} Q^a_i \]

(14)

Note that it also includes orders that will be ordered at future moments and arrive before the current order. For these orders the decision on \( Q^a_i \) has not yet been taken, so it has to be
forecasted on the current ordering moment. We can reformulate the improved ordering policy in terms of the ordering moment by combining (13) and (14):

\[
Q^a_i = \hat{D}^0_{a_i, a_i+1 - a_i} + M_{a_i+1} - \left( I_{a_i} - \hat{D}^0_{a_i, a_i+1 - a_i} + \sum_{t:a_i \geq \eta \land a_i < a_i} Q^a_t \right)
\]

and obtain:

\[
Q^a_i = \hat{D}^0_{a_i, a_i+1 - a_i} + M_{a_i+1} - \left( I_{a_i} + \sum_{t:a_i \geq \eta \land a_i < a_i} Q^a_t \right)
\]

The improved ordering policy covers the total amount of items expected to be required \( \hat{D}^0_{a_i, a_i+1 - a_i} + M_{a_i+1} \) and the total amount that becomes available \( I_{a_i} + \sum_{t:a_i \geq \eta \land a_i < a_i} Q^a_t \), where all arriving orders during the interval \([a_i, ..., a_i]\) are taken into account.

Note the difference between the standard and improved ordering policy. The standard ordering policy computes the inventory position by adding the complete set of already ordered \((t: a_i < o_j)\) but not yet received \((t: \eta \geq o_j)\) orders to the net on hand inventory, while the improved ordering policy states we should add the set of orders that are to be received before the arrival of the current order \((t: a_i < o_j)\), but are not yet received \((t: a_i \geq o_j)\). Only as long as \(A \equiv R\), we have \(a_i < o_j \iff r_i < r_j\), and the outcome of both system state variables is identical: \(\sum_{t:a_i < \eta \land a_i < a_i} Q_i = \sum_{t:a_i \geq \eta \land a_i < a_i} Q_i\). However, in case of expected order crossovers, \(A \neq R\), and the outcome of both system state variables differs.

3 COMPONENTS OF THE IMPROVED ORDERING POLICY

The improved ordering policy (13) aims for each end-of-period inventory \(\hat{I}_{a_i}^{a_i-1, L_i-1} = M_{a_i}\), so it can be rewritten as:

\[
Q^a_i = \hat{D}^0_{a_i, a_i+1 - a_i} + (M_{a_i+1} - M_{a_i}) - (\hat{I}_{a_i}^{a_i-1, L_i-1} - \hat{I}_{a_i}^{a_i-1, L_i-1})
\]

The last term of this expression can further be rewritten.

We can use (14): \(\hat{I}_{a_i}^{a_i-1, L_i-1} = I_{a_i-1} - L_i - D_{a_i-1, L_i, L_i} + \sum_{t:a_i \geq \eta \land a_i < a_i} Q^a_t\),

rewrite (5) as: \(I_{a_i-1} = I_{a_i-1} - L_i - D_{a_i-1, L_i, L_i} + \sum_{t:a_i \geq \eta \land a_i < a_i} Q^a_t\), and obtain:

\[
\hat{I}_{a_i}^{a_i-1, L_i-1} - \hat{I}_{a_i}^{a_i-1, L_i-1} = \left( \left( \hat{D}^{a_i-1, L_i-1}_{a_i-1, L_i-1} - D_{a_i-1, L_i, L_i} + (\hat{D}^{a_i-1, L_i-1}_{a_i-1, L_i, L_i} - D_{a_i-1, L_i, L_i}) \right) + \left( \hat{D}^{a_i-1, L_i-1}_{a_i-1, L_i, L_i} - D_{a_i-1, L_i, L_i} \right) \right)
\]

The improved ordering policy is then rewritten as:

\[
Q^a_i = \hat{D}^0_{a_i, a_i+1 - a_i} + (M_{a_i+1} - M_{a_i}) - \left[ \left( \hat{D}^{a_i-1, L_i-1}_{a_i-1, L_i-1} - D_{a_i-1, L_i-1} + (\hat{D}^{a_i-1, L_i-1}_{a_i-1, L_i, L_i} - D_{a_i-1, L_i, L_i}) \right) + \left( \hat{D}^{a_i-1, L_i-1}_{a_i-1, L_i, L_i} - D_{a_i-1, L_i, L_i} \right) \right]
\]

We will now analyze the tree components of expression (18). For this we use an illustrative example of a crossover order situation introduced in Figure 2. This figure gives an example of a situation with an order crossover \((A \neq R)\) and three order-size decisions. The first order is placed at time \(o_3\) and arrives at \(a_1\). The size of the order controls the inventory level just before \(a_2 = r_3\). The secondly placed order, \(Q^a_3\), is issued at time \(o_2\) but arrives later than the next order that is issued, order \(Q^a_2\). The second and third order cross. The order of the arrival moments is \(\eta < r_3 < r_2 < r_4\).
The first term in expressions (16) and (18) is the demand coverage component. The improved ordering policy covers for the decision on $a_j = a_i - L_i$ demand over a non-negative interval of two consecutive arrival moments $[a_i, a_{i+1}] = [r_j, a_{i+1}]$, and per definition we have $a_{i+1} \geq r_j$. The next decision on $a_{j+1}$ whose order size arrives at $r_{j+1}$ will cover demand over the interval $[r_{j+1}, r_j]$, and the earlier decision on $a_{j-1}$ that arrives at $r_{j-1}$ has covered demand over the interval $[r_{j-1}, r_{j+1}]$.

The second term corrects for a safety level change, also denoted as an order-up-to level change. The improved ordering policy states that the change between the two consecutive arrival moments $a_i$ and $a_{i+1}$ should be covered. Note that the standard ordering policy covers safety level change between $r_j$ and $r_{j+1}$, but $a_{i+1}$ need not be equal to $r_{j+1}$, as can be seen in Figure 2, where for $j=2$: $a_{j+1} = a_4$, while $r_{j+1} = r_3 = a_2$. Hence, the standard ordering policy covers safety level changes between non-consecutive arrival moments, resulting in not achieving the planned inventory levels at intermediate arrival moments. At later decision moments, these errors are corrected, but in the meanwhile there is either too much or too less protection against unforeseen events.

Finally, the third term in (16) and (18) compensates for forecast errors. This type of compensation can be distinguished into two components that have been worked out in (18): (1) a correction for forecast errors that have become apparent between the preceding ordering moment and the current one, and (2) a correction due to the fact that improved forecasts for future demand have become available.

The improved ordering policy only corrects for type 1 forecast error as long as $a_{i-1} - L_{i-1} < a_i - L_i$, i.e. the ordering moment of the order that precedes the arrival of the current order should precede the ordering moment of the current order. Then, the improved ordering policy corrects for all forecast errors that have become apparent between both ordering moments, as can be seen in (18).

However, if $a_{i-1} - L_{i-1} > a_i - L_i$ and noting that $a_{i-1} < a_i$, the forecast error compensation component in the improved ordering policy $I_{a_i} - I_{a_{i-1}} - L_{i-1}$ is unknown at the time of ordering the lot that arrives at $a_i$, as $a_{i-1} - L_{i-1}$ is a future moment. The best we can do is to use an unbiased forecast this error compensation component at time $a_i - L_i$, which

Figure 2 Example of crossover order

The first term in expressions (16) and (18) is the demand coverage component. The improved ordering policy covers for the decision on $a_j = a_i - L_i$ demand over a non-negative interval of two consecutive arrival moments $[a_i, a_{i+1}] = [r_j, a_{i+1}]$, and per definition we have $a_{i+1} \geq r_j$. The next decision on $a_{j+1}$ whose order size arrives at $r_{j+1}$ will cover demand over the interval $[r_{j+1}, r_j]$, and the earlier decision on $a_{j-1}$ that arrives at $r_{j-1}$ has covered demand over the interval $[r_{j-1}, r_{j+1}]$.

The second term corrects for a safety level change, also denoted as an order-up-to level change. The improved ordering policy states that the change between the two consecutive arrival moments $a_i$ and $a_{i+1}$ should be covered. Note that the standard ordering policy covers safety level change between $r_j$ and $r_{j+1}$, but $a_{i+1}$ need not be equal to $r_{j+1}$, as can be seen in Figure 2, where for $j=2$: $a_{j+1} = a_4$, while $r_{j+1} = r_3 = a_2$. Hence, the standard ordering policy covers safety level changes between non-consecutive arrival moments, resulting in not achieving the planned inventory levels at intermediate arrival moments. At later decision moments, these errors are corrected, but in the meanwhile there is either too much or too less protection against unforeseen events.

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The improved ordering policy only corrects for type 1 forecast error as long as $a_{i-1} - L_{i-1} < a_i - L_i$, i.e. the ordering moment of the order that precedes the arrival of the current order should precede the ordering moment of the current order. Then, the improved ordering policy corrects for all forecast errors that have become apparent between both ordering moments, as can be seen in (18).

However, if $a_{i-1} - L_{i-1} > a_i - L_i$ and noting that $a_{i-1} < a_i$, the forecast error compensation component in the improved ordering policy $I_{a_i} - I_{a_{i-1}} - L_{i-1}$ is unknown at the time of ordering the lot that arrives at $a_i$, as $a_{i-1} - L_{i-1}$ is a future moment. The best we can do is to use an unbiased forecast this error compensation component at time $a_i - L_i$, which
The improved ordering policy delays type (1) forecast error compensation until a future moment in time if \( a_{i-1} - L_{i-1} > a_i - L_i \).

The second type of forecast error compensates for improved forecasts that have become available since the former ordering moment. As long as \( a_{i-1} - L_{i-1} \leq a_i - L_i \), compensation should be performed at the current ordering moment when new information has become available. However, if at ordering moment \( a_j = a_i - L_i \) it concludes that a future ordering moment \( a_{j+1} = a_{i-1} - L_{i-1} \) results in an earlier arrival, no forecast error compensation is performed at \( a_j \). The reason is that \( \hat{D}_{a_i - L_i, a_j}^{a_{i-1} - L_{i-1}} \) is not yet known, and the best we can do is to expect it can be forecasted as \( \hat{D}_{a_i - L_i, a_j}^{a_{i-1} - L_{i-1}} = \hat{D}_{a_i - L_i, a_j}^{a_{i-1} - L_{i-1}} \). Therefore, the improved policy states that due to the crossover, delaying the decision to correct for the forecast errors is more effective. At the next ordering moment \( a_{j+1} = a_{i-1} - L_{i-1} \) the type (2) forecast error \( \left( \hat{D}_{a_{j+1}; r_j; t_j} - \hat{D}_{a_{j+1}; r_j; t_j} \right) \) is being corrected, as well as the type (1) forecast error over the interval \([a_{j-1}, a_{j+1}]\).

This decision to delay the forecast error compensation is correct, but the question remains whether all available information has been used in correcting or not. The next subsection shows that our improved ordering policy ignores some information that can be valuable in order to decide on the lot size in a crossover situation.

### 3.1 Information not used by the improved ordering policy

The improved ordering policy neglects to correct for type (2) forecast errors that are behind the demand period that this decision should cover. In case of no crossovers, this is not a problem as a future decision will always take a larger horizon into account. However, in case of order crossovers, an earlier decision might have considered a longer horizon than the next decision. In order to understand this, we refer to Figure 2. Suppose the current moment is \( a_j \) and we have to decide on \( Q_{r_2}^* \) that is expected to arrive on \( a_2 = r_3 \). The former order \( Q_{r_j}^* \) that already has been issued and is expected to arrive later than the current order has forecasted demand over the interval \([r_2, r_3]\), and forecasted the inventory position at \( r_2 \) as well. At the current ordering moment \( a_3 \) new forecasts are available, both on the inventory position at the future moment \( r_2 \) and on the expected demand over the interval \([r_2, r_3]\). The improved ordering policy corrects at ordering moment \( a_{j+1} = a_3 \) for the changes in the forecasts

\[
\hat{D}_{a_j + t_j; r_j; t_j} - \hat{D}_{a_j + t_j; r_j; t_j} = \left( \hat{D}_{a_j + t_j; r_j; t_j} - \hat{D}_{a_j + t_j; r_j; t_j} \right) \quad \text{for the remaining period of the decision taken at } a_1.
\]

Note however that the decision at \( a_3 \) does not completely update the not yet realized part of the forecast \( \hat{D}_{r_j; r_3} \) with the newly information available \( \hat{D}_{a_3; r_3} \). The first term of (18) only corrects for \( \hat{D}_{a_3; r_3} - \hat{D}_{a_3; r_3} \) and the second term of (18) only corrects for \( \hat{D}_{a_3; r_3} - \hat{D}_{a_3; r_3} \), not for \( \hat{D}_{a_3; r_3} - \hat{D}_{a_3; r_3} \).

The second term of (18) should therefore compensate for \( \hat{D}_{a_3; r_3} - \hat{D}_{a_3; r_3} \) as well as \( \hat{D}_{a_3; r_3} - \hat{D}_{a_3; r_3} \). Hence, the improved ordering policy is still heuristic in nature, as it does not use all relevant information available at the moment of ordering. A first improvement is to correct also for changes in the forecast for the interval \([r_j, r_{j+2}]\). However, note that such a correction aims to improve the control on inventory position \( I_{r_{j+2}} = I_{a_{j+2}} \), while at the same time loosening control on the inventory position \( I_{r_j} = I_{a_{j+1}} \).
3.2 Reconsidering the assumption of separability

The consequences of order crossovers have been studied under the assumption that the original optimization problem as stated in (10) can be reduced to a set of optimization problems that can be solved independently based on the most recent information available. This assumption will now be reconsidered.

An optimal policy \( \{Q_1, \ldots, Q_T\} \) minimizes \( G_t(I_{a_t}) \) with functional equation (finite horizon):

\[
G_{J-k}(I_{a_{J-k}}) = g_{J-k}(I_{a_{J-k}}) + G_{J+1-k}(I_{a_{J+1-k}}) \quad \forall k = 1..J-1
\]

terminal condition

\[
G_J(I_{a_J}) = E[V[I_{a_J} - M_{a_J}]]Y^J
\]

where

\[
g_{J-k}(I_{a_{J-k}}) = \min_{Q_{J-k}} E[V[I_{a_{J-k}} - M_{a_{J-k}}] + E_{Q_{J-k}}(g_{J+1-k}(I_{a_{J+1-k}}))]Y^{J-k}
\]

Reduction of

\[
g_{J-k}(I_{a_{J-k}}) = \min_{Q_{J-k}} E[V[I_{a_{J-k}} - M_{a_{J-k}}] + E_{Q_{J-k}}(g_{J+1-k}(I_{a_{J+1-k}}))]Y^{J-k}
\]

to

\[
g_{J-k}(I_{a_{J-k}}) = c + \min_{Q_{J-k}} E[V[I_{a_{J-k}} - M_{a_{J-k}}]Y^{J-k}]
\]

with \( c \) constant, is possible if, by conditioning on the observational history \( Y^{J-k} \) that includes all decisions earlier in time:

(#1) only the current decision \( Q_{J-k} \) affects the level \( I_{a_{J-k}} \)

(#2) the size of \( Q_{J-k} \) does not affect the demand estimation procedure

(#3) the resulting error term \( I_{a_{J-k}} - M_{a_{J-k}} \) consists of white noise.

If these three conditions hold, we can apply separation (Whittle, 1982).

If \( A = R \), the crossover order makes that conditions (#1) and (#3) no longer hold. The presence of a crossover order makes it possible to control several future inventory levels through one decision, causing \( E_{Q_{J-k}}(g_{J+1-k}(I_{a_{J+1-k}})) \) to depend on the current decision. We therefore have to consider the effect of the current decision on these future levels as well.

Let us first consider example Figure 2. On moment \( a_3 \), when deciding on order \( Q_3 \), we control inventory level \( I_{a_3} \), but also inventory level \( I_{a_4} \). Order \( Q_3 \) has been issued before order \( Q_3^* \), controls \( I_{a_3} \) only and will have aimed at \( I_{a_3}^* = M_{a_3} \). When later on deciding on \( Q_3^* \), and based on new information available at time \( a_3 \), we can affect both \( I_{a_3}^* = M_{a_3} \) and \( I_{a_4}^* = M_{a_4} \), as discussed at the end of section 3.1. Therefore, in case of order crossovers a choice has to be made on the weights of these levels.

In order to take the effect of a decision on several moments in future into account, we propose a summation formula over all remaining future levels. Reformulating \( g_{J-k}(I_{a_{J-k}}) \) into

\[
g_{J-k}(I_{a_{J-k}}) = \min_{Q_{J-k}} E\left[ \left. \sum_{t=0}^{k} V[I_{a_{J-t}} - M_{a_{J-t}}]\right| Y^{J-k} \right]
\]

yields a formulation that takes the effect on future inventory levels into account. This formulation closely resembles the original problem (4).

Let \( V \) be a quadratic cost function. Such a cost function is according to the conditions in Section 2. We will work out the decision rule that emerges from (19). At \( a_3 \) we minimize

\[
\min_{Q_3} E[V[I_{a_3} - M_{a_3}] + V[I_{a_3} - M_{a_3}]Y^3] = E[(I_{a_3} - [M_{a_3} + D_{a_3}^{opt} - a_3 - Q_3]^2 + (I_{a_3} - M_{a_3})^2Y^3].
\]

By normalizing the weights we minimize:
\[
\min_{Q^c} \left[ \frac{1}{2} (I_{a_1} - [M_{a_1} + D_{a_1, a_1}^{act} - Q^c])^2 + \frac{1}{2} (I_{a_2} - M_{a_2})^2 \right] Y^3
\]

(20)

using \( \beta (x - a)^2 + (1 - \beta)(x - b)^2 = (x - \{\beta a + (1 - \beta) b\})^2 - \beta (1 - \beta)(a - b)^2 \) for \( 0 < \beta < 1 \) this is equivalent to:

\[
E\left[ \left( I_{a_1} - \left( \frac{1}{2} [M_{a_1} + D_{a_1, a_1}^{act} - Q^c] + \left( 1 - \frac{1}{2} \right) M_{a_2} \right) \right)^2 \right] \min_{Q^c} \frac{1}{2} \left( \frac{1}{2} [M_{a_1} + D_{a_1, a_1}^{act} - Q^c] + \left( 1 - \frac{1}{2} \right) M_{a_2} \right)^2 \right] \left( I_{a_2} - M_{a_2} \right)^2 Y^3 \]

and we note that \((M_{a_1} + D_{a_1, a_1}^{act} - Q^c) = 0\) is not a function of \( Q^c \). Hence we have to solve:

\[
\min_{Q^c} \left[ E\left[ \left( I_{a_1} - \left( \frac{1}{2} [M_{a_1} + D_{a_1, a_1}^{act} - Q^c] + \left( 1 - \frac{1}{2} \right) M_{a_2} \right) \right)^2 \right] \right] Y^3
\]

(21)

Due to the quadratic function, the solution is: \( \hat{I}_{a_1}^a = \frac{1}{2} [M_{a_1} + D_{a_1, a_1}^{act} - Q^c] + \frac{1}{2} M_{a_2} \) and hence the decision rule \( Q^c \) for emerges:

\[
Q^c = \left( \hat{I}_{a_1}^a - \hat{I}_{a_1}^0 \right) + \hat{D}_{a_2, a_2}^{act} = \frac{1}{2} [M_{a_1} + D_{a_1, a_1}^{act} - Q^c] + \frac{1}{2} M_{a_2} - \hat{I}_{a_1}^0 + \hat{D}_{a_2, a_2}^{act} \]

(22)

Note that the weights in (22) are the result of the quadratic cost function. We could use other normalized weights as well if we prefer to improve control over one of these moments. Rule (13) used full weight on \( M_{a_2} \), but it should depend on the actual cost function.

\( Q^c \) has been determined earlier in time, at ordering moment \( \alpha_2 \). At that moment, we had to solve \( \min_{Q^c} E\left[ \left( I_{a_1} - M_{a_1} \right)^2 \right] Y^2 \), which resulted in \( M_{a_1} = \hat{I}_{a_1}^0 = \hat{I}_{a_1}^0 + Q^c - \hat{D}_{a_2, a_2}^{act} \) and hence \( Q^c = M_{a_1} + D_{a_1, a_1}^{act} - \hat{I}_{a_1}^0 \). The forecast \( \hat{I}_{a_1}^0 \) is equal to \( \hat{I}_{a_1}^0 = \hat{I}_{a_1}^0 + \hat{D}_{a_2, a_2}^{act} \) and we note that it includes a forecast of a decision that is not yet taken. Knowing that this decision will try to minimize \( E\left[ \left( I_{a_1} - \left( M_{a_1} + D_{a_1, a_1}^{act} - Q^c \right) \right)^2 \right] + \left( 1 - \frac{1}{2} \right) \left( I_{a_2} - M_{a_2} \right)^2 Y^3 \) \), we estimate this order size at time \( \alpha_2 \) such that \( \left( \hat{I}_{a_1}^0 - \left( M_{a_1} + D_{a_1, a_1}^{act} - Q^c \right) \right) + \left( I_{a_2} - M_{a_2} \right) = 0 \). The first term is zero, so we will choose \( Q^c \) s.t. \( \left( \hat{I}_{a_1}^0 - M_{a_2} \right) = 0 \). As a result, \( Q^c = M_{a_1} - M_{a_2} + D_{a_2, a_2}^{act} \). We see that this order size does not contain an error correction component. Forecast errors that have become apparent in the interval \([\alpha_1, \alpha_2]\) are not corrected in this decision, as the expression for \( Q^c \) does not contain an update of the estimated inventory level at its arrival moment. Forecast error correction is delayed until ordering moment \( \alpha_3 \), as that order is expected to cross with the current order and will arrive earlier.

An important principle is therefore that forecast errors are only to be corrected if the order is expected to arrive earlier than subsequent orders. The optimal policy needs to look ahead for future orders that may arrive before the current order. The search can be restricted to future ordering moments smaller than the expected arrival moment of the current order.

4 SAFETY STOCKS

The preceding sections assumed that the safety stock levels varied over time and were externally determined. Our motivation for this modeling choice was that this resembled the problem setting that we originally encountered in industry. Section 2 discussed that other authors (i.e. Urban, 2005 and Aviv, 2003) determine the safety stock levels within the model, based on the characteristics of the stochastic processes. The question therefore arises whether we should include the determination of the safety stock levels within the inventory model or not. Our approach allows us to determine characteristics of the distribution of the inventory position at various moments in future, based on the underlying distributions of demand, lead
time, et cetera. In case of expected order crossovers, we assume that lead time uncertainty can be ignored, as the focus is on the dynamic variation in lead times in stead of the stochastic variation. We further assume that information on the demand model is known and that the variation of the random variables in this model can be described using normal distributions. The inventory positions just before the expected arrival of the orders, \( \{ I_n \} \), can now be described as a stochastic process. The cost depend on the expected inventory and the variance of this variable, so both stock out and holding costs can be used.

5 CONCLUSIONS

The standard myopic ordering policy does not take expected order crossovers into account. Gaalman and Riezebos (2005) developed an improved order-up-to policy, based on a stochastic dynamic programming formulation of the originating problem. This improved myopic order-up-to-rule is shown to be heuristic in nature. It does not completely correct for updated demand forecasts that have become available since the last ordering moment. And it is still myopic in nature. It assumes that the originating stochastic dynamic programming problem can be reduced to a series of independent optimization problems decisions at each ordering moment. We showed that this assumption does not hold in case of expected order crossovers. We deduced a new generalized ordering policy from a modified stochastic dynamic programming problem. This generalized policy takes into account that one decision can affect (or control) several future inventory positions. Future research should be directed towards further insights in this inventory policy, specifically on the issue of safety stock determination from within the model.

6 REFERENCES


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